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# Analytical relationships between metric and centrality measures of a network and its dual

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ABSTRACT

The centrality and efficiency measures of a network $G$ are strongly related to the respective measures on the dual $G^{\star}$ and the bipartite $B(G)$ associated networks. We show some relationships between the Bonacich centralities $c(G)$, $c(G^{\star})$ and $c(B(G))$ and between the efficiencies $E(G)$ and $E(G^{\star})$ and we compute the behavior of these parameters in some examples.



## 1. Introduction and notation

Complex networks are used for modeling different systems of the real world, such as Internet, neural networks, metabolic and protein networks, social networks and the World Wide Web [1–5]. These systems are known to have behavioral and structural characteristics in common, and they can be studied by using non-linear mathematical models and computer modeling approaches. From a mathematical point of view, these objects have been classically studied in the realm of graph theory, but due to the complexity of such objects derived from the size and the dynamics on them, new tools are required, shaping the scientific area known as *complex network analysis*, that involves not only mathematical tools (including probability, dynamical system analysis, graph theory, matrix analysis and others), but also techniques coming from other fields (as statistical mechanics or computer sciences, to name a couple of them).

The motivation behind this note is to consider the importance that edges have sometimes over nodes in the context of networks and graphs. An example of this comes from urbanism where the line (dual) graph, $G^{\star}$ (see below for definition), associated to a given graph, $G = (V, E)$, representing a given network is considered [6,7]. Distribution networks constitute another example of this situation. The following natural question arises: what relations (if any) can be established between the properties of $G$ and $G^{\star}$? The idea is that sometimes it might be simpler to work with $G^{\star}$ than with the initial graph $G$, or conversely, and having estimations of the parameters of one of the graphs by means of the corresponding ones in the other graph can be helpful.

In particular, we are interested in providing analytical relations between some parameters associated to $G$ and $G^{\star}$. In doing so the introduction of the bipartite graph, $B(G)$, associated to $G = (V, E)$ will be of help (see below for definition).

There are many different parameters that measure different properties related to the network performance, but in this note we will only consider the Bonacich centrality (based on the eigenvectors associated to the spectral radius) [8,9] and the efficiency of a network [10–13].

In order to investigate such properties, it is necessary to understand the main structure of the underlying network [1,4] and also to care about other topological aspects which complement it.

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From a schematic point of view, a complex network is a mathematical object $G = (V, E)$ composed by a set of nodes or vertices $V = \{1, \dots, n\}$ that are pairwise joined by links or edges $\{\ell_1, \dots, \ell_m\}$. We consider the adjacency matrix $A(G) = (a_{ij})$ determined by the conditions

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{if } \{i, j\} \notin E. \end{cases}$$

The bipartite network $B(G)$ associated to $G$ is defined by $B(G) = (X \cup E, E(B(G)))$ whose adjacency matrix is given by

$$A(B(G)) = \left( \begin{array}{c|c} 0 & I(G) \\ \hline I(G)^t & 0 \end{array} \right),$$

where $I(G) = I_G = (I_{ij})$ is the incidence matrix of $G$ defined by

$$I_{ij} = \begin{cases} 1 & \text{if edge } \ell_j \text{ is incident with node } i \\ 0 & \text{otherwise.} \end{cases}$$

It is shown that

$$A(B(G))^2 = \left( \begin{array}{c|c} A(G) + gr & 0 \\ \hline 0 & A(G^\star) + 2I_m \end{array} \right),$$

where $A(G) + gr = I_G I_G^t$ denotes the matrix obtained by adding to $A(G)$ the diagonal matrix $(b_{ij})$ where $(b_{ii})$ is the degree of the vertex $i$, and $G^\star$ denotes the line (or dual) network associated to $G$ [14, page,26]. Recall that the line graph associated to $G = (V, E)$ is the network $G^\star = (E, L)$ whose set of nodes is the initial set of edges of the graph $G$, with the assumption that two such nodes $\ell_i$ and $\ell_j$ are connected by the edge $\{\ell_i, \ell_j\}$ if on the initial graph $G$ the edges $\ell_i$ and $\ell_j$ share some node. Observe that the equality $I_G^t I_G = A_{G^\star} + 2I_m$, where $I_m$ is the identity matrix in $\mathbb{R}^m$, trivially holds.

If we know the Bonacich centrality $c(G^\star)$, we can recover $c(B(G))$ and reciprocally. If, in addition, $G$ is regular then each of the three centralities can be recovered from any of the other. On the other hand the efficiencies of the dual graph $G^\star$ and the primal graph $G$ are estimated by means of inequalities. One of them is easy to prove and is given by $E(G^\star) \geq C_{n,m} E(G)$, where $C_{n,m}$ is an absolute constant only depending on the number of nodes and links, while the other requires some extra work.

## 2. Relationships among the centrality measures

This section is devoted to present some relations between the Bonacich centralities of $G$, $G^\star$ and $B(G)$, as announced. Let us recall that the Bonacich centrality of a complex network $G$ is the non-negative normalized eigenvector $c_G \in \mathbb{R}^n$ associated to the spectral radius of the transposed adjacency matrix of $G$ [8,9,4].

**Theorem 2.1.** *Let $G = (V, E)$ a connected and non-directed graph with $n$ vertices and $m$ edges. Let $c_G \in \mathbb{R}^n$, $c_{G*} \in \mathbb{R}^m$ and $c_{B(G)} = (c_1, c_2) \in \mathbb{R}^n \times \mathbb{R}^m$, the Bonacich centralities of $G$, $G*$ and $B(G)$. Then*

(i) $c_{G^\star} = \frac{c_2}{\|c_2\|_1}$. *In addition to this, if $G$ is regular, then* $c_G = \frac{c_1}{\|c_1\|_1}$.

(ii) *Reciprocally* $c_{B(G)} = \frac{1}{2}(c_G, c_{G^\star})$ *and*

$$c_G = \frac{I_G\, c_{G^\star}}{\|I_G\, c_{G^\star}\|_1}, \qquad c_{G^\star} = \frac{I_G^t\, c_G}{\|I_G^t\, c_G\|_1},$$

*where* $\|v\|_1 = \sum_{i=1}^n |v_i|$ *for any arbitrary* $v = (v_1, \dots, v_n) \in \mathbb{R}^n$.

**Proof.** (i) Let $c_{B(G)} > 0$ the Bonacich centrality of $B(G)$ (which exists if $G$ is connected). We can think of $c_{B(G)}$ as a vector $(c_1, c_2) \in \mathbb{R}^n \times \mathbb{R}^m$ with $c_1, c_2 > 0$. Clearly $c_1$ and $c_2$ are positive eigenvectors of $I_G I_G^t$ and $I_G^t I_G$ respectively. Since $I_G^t I_G = A_{G*} + 2I_m$ it follows that $c_2$ is eigenvector of $A_{G*}$ and $c_2 = c_{G*}\|c_2\|_1$.

Analogously

$$I_G I_G^t = A_G + \begin{pmatrix} gr(1) & 0 & & & 0 \\ 0 & . & & & \\ & & . & & \\ & & & . & 0 \\ 0 & & & 0 & gr(n) \end{pmatrix}$$

and if suppose in addition that $G$ is $k$-regular for some $k \in \mathbb{N}$, then this equality becomes $I_G I_G^t = A_G + kI_n$. Hence $c_1 > 0$ is an eigenvector of $A_G$ and $c_1 = c_G\,.\|c_1\|_1$.

(ii) Let $c_{B(G)} \in \mathbb{R}^{n+m}$ be the Bonacich centrality of $B(G)$, that is the normalized positive vector associated to the eigenvalue $\rho(A_{B(G)})$. Since $A_{B(G)}$ is symmetric (note that $B(G)$ is non-directed being $G$ non-directed), then it is diagonalizable. Consider the eigenvalues $\{\lambda_1, \dots, \lambda_{n+m}\}$ counted according to their multiplicity. Renaming if needed we can assume without loss of generality that

$$\rho(A_{B(G)}) = |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n+m}|.$$

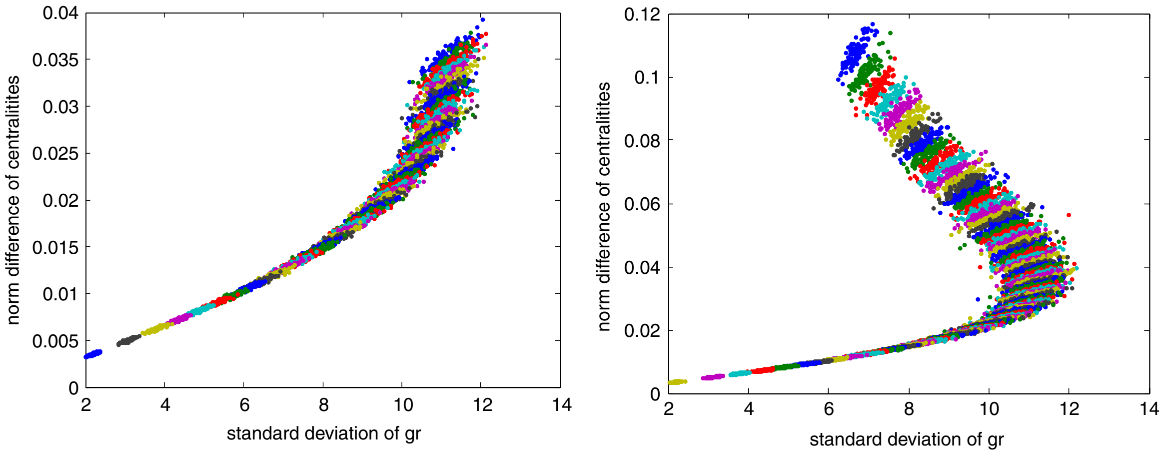


**Fig. 1.** A computational comparison between the standard deviation of the degree vector and the Bonacich centralities of $G + gr$ and $G$, where $G$ is an Erdős–Renji random network of 500 nodes and linking probabilities $0.5 \leq p \leq 0.99$ (on the left) and $0.1 \leq p \leq 0.99$ (on the right).

Consider a corresponding base of eigenvectors $\{v_1, \dots, v_{n+m}\}$, where each $v_i = (u_i, w_i) \in \mathbb{R}^{n+m}$ is associated to $\lambda_i$. From $\lambda_i v_i = A_{B(G)} v_i$ it readily follows that $\lambda_i u_i = I_G w_i$ and

$$\begin{cases} \lambda_i u_i = I_G w_i \\ \lambda_i w_i = I_G^t u_i. \end{cases} \tag{$*$}$$

Also, if $\lambda_i \neq 0$ then $u_i \neq 0$ and $w_i \neq 0$. Indeed,assume that $u_i = 0$. Then, from $(*)$ follows that $0 = I_G^t u_i = \lambda_i w_i$ and hence that $w_i = 0$ by the assumption on $\lambda_i$. Then $v_i = (0, 0) = 0 \in \mathbb{R}^{n+m}$ which is impossible since $v_i$ is an eigenvector. Analogously $w_i \neq 0$.

Let us check now that $u_i$ and $w_i$ are eigenvectors of $I_G I_G^t$ and $I_G^t I_G$ respectively. Indeed, notice that

$$I_G I_G^t u_i = I_G(I_G^t u_i) = I_G(\lambda_i w_i) = \lambda_i^2 u_i,$$
$$I_G^t I_G w_i = I_G^t(I_G w_i) = I_G^t(\lambda_i u_i) = \lambda_i^2 w_i,$$

for all $i = 1, \dots, n + m$. In other words, the set $\{\lambda_1^2, \dots, \lambda_{n+m}^2\}$ is formed by eigenvalues of both $I_G I_G^t$ and $I_G^t I_G$, since if $\lambda$ is an eigenvalue of a *bipartite* matrix

$$B = \left(\begin{array}{c|c} 0 & C \\ \hline D & 0 \end{array}\right),$$

then $-\lambda$ is also an eigenvalue.

We claim in fact that the set $\{\lambda_1^2, \dots, \lambda_{n+m}^2\}$ contains all possible eigenvalues of both $I_G I_G^t$ and $I_G^t I_G$. Indeed, let $\mu \in \mathbb{R}$ and $0 \neq v \in \mathbb{R}^n$ such that $\mu v = I_G I_G^t v$; then taking $\overline{v} = (v, 0) \in \mathbb{R}^{n+m}$ we get

$$A_{B(G)}^2 \overline{v} = \begin{pmatrix} I_G I_G^t v \\ 0 \end{pmatrix} = \mu \begin{pmatrix} v \\ 0 \end{pmatrix} = \mu \overline{v},$$

and hence $\mu$ is an eigenvalue of $A_{B(G)^2}$. Consequently the spectra of $I_G I_G^t$ and $I_G^t I_G$ are included in $\{\lambda_1^2, \dots, \lambda_{n+m}^2\}$. $\square$

Based on a continuity argument we conjecture that the discrepancy between the Bonacich centralities of $G + gr$ and $G$ should narrow as the regularity of $G$ increases. In fact, it is straightforward to check that if $G$ is regular then both centralities coincide and some simulations seem to support this conjecture. In Fig. 1, the standard deviation of the degree vector vs. $\|c(G) - c(G + gr)\|_1$ for a random test of 500 nodes Erdős–Renji networks is plotted and the strong correlation between these two magnitudes is shown. With this approach it seems that some measure of the irregularity of the graph should be required to make things precise (see, for example, [15,16]). This will be addressed somewhere else.

## 3. Relationships between metric properties of $G$ and $G^\star$

In this section we are interested in finding relations between the efficiency of the graph $G = (V, E)$, the line graph $G^\star$ and the bipartite graph $B(G)$. We start with the relation between $E(G)$ and $E(G^\star)$. Recall that the efficiency of a complex network $G$ is the value

$$E(G) = \frac{1}{n(n-1)} \sum_{i \neq j} \frac{1}{d_G(i, j)},$$

where $d_G(i,j)$ is the distance between nodes $i$ and $j$. An analogous expression for the efficiency in the line graph $G^\star$ requires a previous understanding of the meaning of the distance of edges in $G$ in order to give a distance of nodes in $G^\star$. The following result is well known [14, page,302].

**Proposition 3.1.** *Let $i,j$ and $i',j'$ be two pair of nodes in $G=(V,E)$ joined respectively by the edges $\ell=\{i,j\}$ $\ell'=\{i',j'\}$ and such that $\ell\neq\ell'$. Then, the distance in $G^\star=(E,L)$ between the edges $\ell$ and $\ell'$ is $d_{G*}(\ell,\ell')=1+d_G(\{i,j\},\{i',j'\})$, where $d_G$ is the Hausdorff distance between the sets $\{i,j\}$ and $\{i',j'\}$, i.e.*

$$d_{G^\star}(\ell,\ell')=1+\min\{d_G(i,i'),d_G(i,j'),d_G(j,i'),d_G(j,j')\}.$$

We can now give a meaning to the efficiency of $G^\star$ by means of the usual formula

$$E(G^\star)=\frac{1}{m(m-1)}\sum_{\ell\neq\ell'}\frac{1}{d_{G^\star}(\ell,\ell')}.$$

Similarly, if we want to define the efficiency in the bipartite graph $B(G)$ we need first an expression for the distance in $B(G)$. This is the content of the next lemma whose proof is straightforward.

**Lemma 3.2.** *Let $B(G)$ the bipartite graph associated to $G=(V,E)$. Then for every $i,j,k,i',j'\in V$*

(i) $d_{B(E)}(i,j)=2d_G(i,j)$,
(ii) $d_{B(G)}(i,\{j,k\})=1+2\min\{d_G(i,j),d_G(i,k)\}$,
(iii) $d_{B(E)}(\{i,j\},\{i',j'\})=2d_{G^\star}(\{i,j\},\{i',j'\})$.

Now efficiency in $B(G)$ can be expressed as

$$\begin{aligned}E(B(G)) &= \frac{1}{(m+n)(m+n-1)}\sum_{a\neq b\in V\cup E}\frac{1}{d_{B(G)}(a,b)}\\ &= \frac{1}{(m+n)(m+n-1)}\left[\sum_{i,j\in V,\ i\neq j}\frac{1}{d_{B(E)}(i,j)}+\sum_{\ell\neq\ell'}\frac{1}{d_{B(E)}(\ell,\ell')}+2\sum_{i\in V,\ \ell\in E}\frac{1}{d_{B(E)}(i,\ell)}\right].\end{aligned}$$

In order to present our result relating efficiencies of $G$, $G^\star$ and $B(G)$ we need a pair of technical lemmas whose proofs are left to the reader:

**Lemma 3.3.** *Let $a,b,c,d,e,f\in(0,\infty]$. If we consider $x=a+b\min\{c,d\}$ and $y=a+b\min\{c,d,e,f\}$. Then*

(i) $\frac{1}{x}\leq\frac{1}{b}\left(\frac{1}{c}+\frac{1}{d}\right)$,
(ii) $\frac{1}{y}\leq\frac{1}{b}\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right)$.

**Lemma 3.4.** *Let $a,b,c,d,e,f\in(0,\infty]$ such that $1\leq c,d,e,f$. If we consider again $x=a+b\min\{c,d\}$ and $y=a+b\min\{c,d,e,f\}$. Then*

(i) $\frac{1}{x}\geq\frac{1}{2}\frac{1}{a+b}\left(\frac{1}{c}+\frac{1}{d}\right)$,
(ii) $\frac{1}{y}\geq\frac{1}{4}\frac{1}{a+b}\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right)$.

Notice that the estimations in Lemma 3.4 are asymptotically sharp. Indeed, taking $c=d=e=f=n$ in (i), we get that if $n\longrightarrow\infty$, then

$$0<\frac{1}{x}<\frac{1}{b}\left(\frac{2}{n}\right)\longrightarrow 0,$$
$$0<\frac{1}{y}<\frac{1}{b}\left(\frac{4}{n}\right)\longrightarrow 0.$$

Similarly, by taking $c=d=e=f=1$ in (ii) we get that $x=y=a+b$.

**Theorem 3.5.** *Let $G=(V,E)$ and $G^\star=(E,L)$ be as above where $n$ is the number of nodes of $G$, $m$ is the number of nodes $G^\star$ and $p$ is the number of edges of $G^\star$. Then*

$$\frac{n(n-1)}{8m(m-1)}E(G)+\frac{15p-2}{8m(m-1)}\leq E(G^\star)\leq\max_{i\neq j}(gr(i)gr(j))\frac{n(n-1)}{m(m-1)}E(G)+\frac{2p}{m(m-1)}.$$

**Proof.** Let us start with the right hand side inequality. Notice that if $\ell$ and $\ell'$ are two different edges of $G$ such $d_{G^\star}(\ell,\ell')\geq 2$, then by Lemma 3.3(ii) with $y=d_{G^\star}(\ell,\ell')$, $a=1$ and $b=1$ we get

$$\frac{1}{d_{G^\star}(\ell,\ell')}\leq\frac{1}{d_G(i,i')}+\frac{1}{d_G(i,j')}+\frac{1}{d_G(j,i')}+\frac{1}{d_G(j,j')}.$$

Thus

$$\begin{aligned} m(m-1)E(G^\star) &= \sum_{d(\ell,\ell')=1} \frac{1}{d_{G^\star}(\ell,\ell')} + \sum_{d_{G^\star}(\ell,\ell')\geq 2} \frac{1}{d_{G^\star}(\ell,\ell')} \\ &\leq 2p + \sum_{i\neq j} \frac{gr(i)gr(j)}{d_G(i,j)} \leq 2p + \max_{i\neq j}(gr(i)gr(j)) \sum_{i\neq j} \frac{1}{d_G(i,j)} \\ &= 2p + \max_{i\neq j}(gr(i)gr(j))n(n-1)E(G), \end{aligned}$$

which leads to the right hand side inequality.

In order to deal with the remaining inequality observe that we can split the expression of the efficiency for $G$ as follows:

$$n(n-1)E(G) = \sum_{d_G(i,j)=1} \frac{1}{d_G(i,j)} + \sum_{d_G(i,j)=2} \frac{1}{d_G(i,j)} + \sum_{d_G(i,j)\geq 3} \frac{1}{d_G(i,j)}.$$

Notice that the first term of the sum is $2m$ while the second term is bounded by $p$ and the third term is bounded by

$$\sum_{d_{G^\star}(\ell,\ell')\geq 2} \frac{8}{d_{G^\star}(\ell,\ell')},$$

which is a consequence of the fact that

$$\frac{1}{d_{G^\star}(\ell,\ell')} \geq \frac{1}{8}\left[\frac{1}{d_G(i,i')} + \frac{1}{d_G(i,j')} + \frac{1}{d_G(j,i')} + \frac{1}{d_G(j,j')}\right],$$

obtained from Lemma 3.4(ii), taking $y = d_{G^\star}(\ell,\ell')$, $a = 1$ and $b = 1$. Thus,

$$\begin{aligned} n(n-1)E(G) &\leq 2m - 15p + 8\left[2p + \sum_{d_{G^\star}(\ell,\ell')\geq 2} \frac{1}{d_{G^\star}(\ell,\ell')}\right] \\ &= 2m - 15p + 8m(m-1)E(G^\star), \end{aligned}$$

which proves that

$$\begin{aligned} E(G^\star) &\geq \frac{n(n-1)E(G) - 2m + 15p}{8m(m-1)} \\ &= \frac{n(n-1)}{8m(m-1)}E(G) + \frac{15p}{8m(m-1)} - \frac{1}{4(m-1)}. \quad \square \end{aligned}$$

Notice that if instead of in terms of the maximum degrees of the nodes of $G$ we want to give the estimations above in terms of $n$, $m$ and $p$ only we can use known fact that

$$p = \frac{1}{2}\left(\sum_{i\in V} gr(i)^2\right) - m \geq \frac{4m^2}{2n} - m,$$

which is derived from Cauchy–Schwarz inequality. On the other hand it is equally known that

$$\sum_{i\in V} gr(i)^2 \leq \frac{2m^2}{n-1} + m(n-2),$$

for all $n \geq 2$ (see [17]).

**Theorem 3.6.** *Let $G = (V, E)$, $G^\star = (E, L)$ and $B(G)$ be as above where $n$ is the number of nodes of $G$ and $m$ is the number of nodes $G^\star$. Then*

$$\frac{n(n-1)E(G)}{2(n+m)(n+m-1)} + \frac{m(m-1)E(G^\star)}{2(n+m)(n+m-1)} + \frac{4m}{(n+m)(n+m-1)} \leq E(B(G)),$$

$$E(B(G)) \leq \frac{5}{2}\frac{n(n-1)E(G)}{(n+m)(n+m-1)} + \frac{1}{2}\frac{m(m-1)E(G^\star)}{(n+m)(n+m-1)} + \frac{4m}{(n+m)(n+m-1)}.$$

**Proof.** We proceed as in the proof of Theorem 3.5 by splitting now the expression of the efficiency of $B(G)$ in three summands, as above, corresponding to the three manners in which the distance between nodes in $B(G)$ can be obtained, namely, by comparing nodes of $G$, or edges of $G$ or nodes and edges of $G$. Then we use Lemma 3.2 to conclude the result. $\square$

## 4. Conclusions

We have shown that the efficiency and centrality of a network *G* are strongly correlated to the corresponding values of its dual $G^\star$. The analytical results presented confirm the relations conjectured in [6,7] concerning the structural properties of a network and its dual, not only in the particular case of urban street networks but also in the general case.

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